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Uniqueness of five-dimensional supersymmetric black holes

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Abstract: A classification of supersymmetric solutions of five dimensional ungauged supergravity coupled to arbitrary many abelian vector multiplets is used to prove a uniqueness theorem for asymptotically flat supersymmetric black holes with regular horizons. It is shown that the near-horizon geometries of solutions for which the scalars and gauge field strengths are sufficiently regular on the horizon are flat space, $AdS_3 \times S^2$, or the near-horizon BMPV solution. Furthermore, the only black hole which has the near-horizon BMPV geometry for its near-horizon geometry is the solution found by Chamseddine and Sabra.

Keywords: Black Holes in String Theory, Supergravity Models.

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1. Introduction

In the past few years there has been a renewed interest in examining various aspects of supergravity theories and their solutions. Considerable progress has been made in our understanding of the structure of supersymmetric solutions of several supergravity theories. In particular, by generalising the methods originally used by Tod to analyse certain four-dimensional supergravities [1, 2] it has been possible to construct classifications of supersymmetric solutions of several supergravities in four, five and six dimensions [3]–[6] which provide a comprehensive description of such solutions (although the 1/2 and (possibly) 3/4 supersymmetric solutions which we expect to occur in the minimal gauged five-dimensional supergravity [4] have yet to be fully analysed).

Rather less is known about the structure of solutions in higher dimensional supergravities. For example, although all maximally supersymmetric solutions of eleven dimensional supergravity are known [7], and all solutions preserving 1/32 of supersymmetry have been classified [8, 9]; solutions preserving intermediate proportions of supersymmetry have yet to be systematically classified. Nevertheless, one can still use similar methods to examine certain restricted classes of higher dimensional solutions; some recent examples of such analysis are [10] and [11]. All of this activity has produced a very large number of new supergravity solutions, many of which have interesting properties. It is clear that considerable work remains to be done in this area.

One particularly useful application of the lower dimensional classifications is in black hole physics. This is of some interest, because of the important role which string theory has played in our understanding of the microscopic origin of the entropy of certain four and five-dimensional black holes [12]–[18]. In the case of five-dimensional black holes, the derivation of the entropy relies on establishing a correspondence between black holes and string states. Clearly, this is more straightforward if there is a black hole uniqueness theorem which constrains the types of possible black hole solutions. However, such uniqueness cannot be taken for granted in five (or more) dimensions, and some explicit black ring solutions have been constructed which violate uniqueness [19, 20]; these solutions do not however preserve any supersymmetry. Although there has been progress in establishing a correspondence between string states and certain types of black ring solutions [21], in general the lack of black hole uniqueness in higher dimensions makes the construction of the black hole/string state correspondence rather more complicated. It has, however, been possible to construct a uniqueness theorem for supersymmetric black hole solutions of the minimal ungauged five-dimensional supergravity [22]. The theorem classifies all near-horizon geometries of solutions which have a regular horizon; the possible regular near-horizon geometries are flat space, $AdS_3 \times S^2$ or the (under-rotating) BMPV near-horizon geometry. In addition, the only solution which has a regular horizon with near-horizon BMPV as its near-horizon geometry is the BMPV black hole [15].

The purpose of this paper is to extend this uniqueness theorem to include supersymmetric black hole solutions of the ungauged five-dimensional supergravity theory coupled to arbitrary many vector multiplets. This extension is useful because, although the minimal theory has many interesting properties, it corresponds to a rather restricted class of higher dimensional solutions. In order to investigate the higher dimensional physics more fully by compactification to lower dimensions, one must typically couple the lower dimensional theory to additional matter. Thus, the extension of the minimal uniqueness theory to the non-minimal theory under consideration here represents a step towards this goal. It is interesting to note that a (partial) classification of solutions of gauged five-dimensional supergravity coupled to arbitrary many vector multiplets constructed in [23] plays a crucial role in the extension of the uniqueness theorem.

The plan of this paper is as follows. In section 2.1 we review some basic properties of the ungauged five-dimensional supergravity coupled to arbitrary many vector multiplets. The classification of solutions constructed in [23] (with vanishing gauge parameter) is summarized in section 2.2. In section 2.3 the black hole solutions of this theory constructed in [24] and [25] are given using the conventions of section 2.2. The extension of the uniqueness theorem is presented in section 3. The theorem is split into two parts. In 3.1, the near-horizon geometry of a black hole with a single regular connected horizon is derived; it is shown that the near-horizon geometry is locally isometric to flat space, $AdS_3 \times S^2$, or the near-horizon geometry of the BMPV solution of the minimal theory. In 3.2 the global properties of the solution with near-horizon BMPV for its near-horizon geometry are examined, and it is shown that the only such solution is that found by Chamseddine and Sabra in [24] and [25]. Some conclusions are given in section 4.

2. Supersymmetric solutions of $\mathcal{N}=1$ supergravity

2.1 $\mathcal{N}=1$ supergravity

The action of $\mathcal{N}=1$ $D=5$ ungauged supergravity coupled to abelian vector multiplets with scalars taking values in a symmetric space is

$$S=\frac{1}{16\pi G}\int\left({}^5R-Q_{IJ}F^I\wedge *F^J-Q_{IJ}dX^I\wedge\star dX^J-\frac{1}{6}C_{IJK}F^I\wedge F^J\wedge A^K\right), \quad (2.1)$$

where we use a positive signature metric and the fermions have been set to zero. I,J,K take values $1\ldots n$ and C_{IJK} are constants that are symmetric on IJK and obey

$$C_{IJK}C_{J'(LM}C_{PQ)K'}\delta^{JJ'}\delta^{KK'}=\frac{4}{3}\delta_{I(L}C_{MPQ)}\,. \quad (2.2)$$

The X^I are scalars which are constrained via

$$\frac{1}{6}C_{IJK}X^IX^JX^K=1\,. \quad (2.3)$$

We may regard the X^I as being functions of $n-1$ unconstrained scalars ϕ^a. It is convenient to define

$$X_I\equiv\frac{1}{6}C_{IJK}X^JX^K \quad (2.4)$$

so that the condition (2.3) becomes

$$X_IX^I=1\,. \quad (2.5)$$

In addition, the coupling Q_{IJ} depends on the scalars via

$$Q_{IJ}=\frac{9}{2}X_IX_J-\frac{1}{2}C_{IJK}X^K\,. \quad (2.6)$$

Additional useful identities which are satisfied as a consequence of the Very Special geometry can be found in [23].

The Einstein equation is given by

$$-{}^5R_{\alpha\beta}+Q_{IJ}F^I{}_{\alpha\lambda}F^J{}_\beta{}^\lambda+Q_{IJ}\nabla_\alpha X^I\nabla_\beta X^J-\frac{1}{6}g_{\alpha\beta}\left(Q_{IJ}F^I{}_{\mu\nu}F^{J\mu\nu}\right)=0 \quad (2.7)$$

the gauge equations are

$$d\left(Q_{IJ}\star F^J\right)=-\frac{1}{4}C_{IJK}F^J\wedge F^K\,, \quad (2.8)$$

and the scalar equation can be written as

$$\begin{aligned}&d\left(\star dX_I\right)-\left(\frac{1}{6}C_{MNI}-\frac{1}{2}X_IC_{MNJ}X^J\right)dX^M\wedge\star dX^N+\\&+\left(X_MX^PC_{NPI}-\frac{1}{6}C_{MNI}-6X_IX_MX_N+\frac{1}{6}X_IC_{MNJ}X^J\right)F^M\wedge\star F^N=0\,. \quad (2.9)\end{aligned}$$

In addition, for a bosonic background to be supersymmetric there must be a spinor ϵ^a. From this Killing spinor we can construct tensors from spinor bilinears, which can be used to classify the general supersymmetric solutions of this theory. This classification was presented in [23] for the solutions of the gauged theory; we shall recap the main results which are somewhat simpler in the case of the ungauged theory. In particular, we obtain a scalar f, a vector V and three 2-forms $J^{(i)}$ which satisfy the algebraic relations

$$
\begin{aligned}
V_\alpha V^\alpha &= -f^2\,, && (2.10)\\
J^{(i)} \wedge J^{(j)} &= -2\delta_{ij} f \star V\,, && (2.11)\\
i_V J^{(i)} &= 0\,, && (2.12)\\
i_V \star J^{(i)} &= -f J^{(i)}\,, && (2.13)\\
J^{(i)}_{\gamma\alpha} J^{(j)\gamma}_{\beta} &= \delta_{ij}\left(f^2\eta_{\alpha\beta} + V_\alpha V_\beta\right) - \epsilon_{ijk} f J^{(k)}_{\alpha\beta}\,, && (2.14)
\end{aligned}
$$

where $\epsilon_{123} = +1$ and, for a vector Y and p-form A, $(i_Y A)_{\alpha_1\ldots\alpha_{p-1}} \equiv Y^\beta A_{\beta\alpha_1\ldots\alpha_{p-1}}$. In addition to these algebraic relations, the bilinears also satisfy differential constraints as a consequence of the gravitino and dilatino equations. These differential constraints were computed for the more general gauged theory in [23], and the equations for the ungauged theory can be obtained from those in [23] by setting the gauge parameter χ to vanish; in particular, we find that V is a Killing vector satisfying

$$
\mathcal{L}_V f = 0\,, \qquad \mathcal{L}_V V = 0\,, \qquad \mathcal{L}_V F^I = \mathcal{L}_V J^i = 0\,. \tag{2.15}
$$

2.2 The timelike case

It is useful to distinguish two cases depending on whether the scalar f vanishes everywhere or not. In the "null case", the vector V is globally a null Killing vector with $f = 0$. As we are interested in investigating the properties of black hole solutions, we shall concentrate on the latter "timelike case" . Take an open set $\mathcal{U}$ in which f is positive and hence V is a timelike Killing vector field. We shall summarize the constraints imposed by supersymmetry in the region $\mathcal{U}$.

Introduce coordinates (t, x^m) such that $V = \partial/\partial t$. The metric can then be written locally as

$$
ds^2 = -f^2(dt + \omega)^2 + f^{-1} h_{mn} dx^m dx^n\,. \tag{2.16}
$$

The metric h_{mn} can be regarded as the metric on a four dimensional riemannian manifold, which we shall refer to as the "base space" B, and ω is a 1-form on B. Since V is Killing, f, ω and h are independent of t. We shall reduce the necessary and sufficient conditions for supersymmetry to a set of equations on B. Let

$$
e^0 = f(dt + \omega)\,. \tag{2.17}
$$

We choose the orientation of B so that $e^0 \wedge \eta_4$ is positively oriented in five dimensions, where η_4 is the volume form of B. The two form $d\omega$ can be split into self-dual and anti-self-dual parts on B:

$$
f d\omega = G^+ + G^-\,, \tag{2.18}
$$

where the factor of f is included for convenience.

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Equation (2.12) implies that the 2-forms $J^{(i)}$ can be regarded as 2-forms on the base space and equation (2.13) implies that they are anti-self-dual:

$$\star_4 J^{(i)} = -J^{(i)}\,, \tag{2.19}$$

where $\star_4$ denotes the Hodge dual on B. equation (2.14) can be written

$$J^{(i)}{}_m{}^p J^{(j)}{}_p{}^n = -\delta^{ij}\delta_m{}^n + \epsilon_{ijk}J^{(k)}{}_m{}^n\,, \tag{2.20}$$

where indices $m, n, \ldots$ have been raised with h^{mn}, the inverse of h_{mn}, so the $J^{(i)}$ satisfy the algebra of imaginary unit quaternions. In addition, from the differential constraints, we find that the J^i are covariantly constant on B and so the base space is hyper-Kähler with hyper-complex structures J^i.

The differential constraints on the bilinears also constrain the gauge field strengths. We find that

$$F^I = d(X^I e^0) + \Theta^I\,, \tag{2.21}$$

where Θ^I is a self-dual 2-form on B satisfying

$$X_I \Theta^I = -\frac{2}{3}G^+\,. \tag{2.22}$$

In fact these conditions are sufficient to ensure the existence of a Killing spinor preserving 4 of the 8 supersymmetries. The Killing spinor ϵ is given by

$$\epsilon = f^{\frac{1}{2}}\epsilon_0\,, \tag{2.23}$$

where ϵ_0 is covariantly constant on the hyper-Kähler base space and satisfies[1]

$$\gamma^0\epsilon = i\epsilon\,. \tag{2.24}$$

However, as we are interested in supersymmetric *solutions* we also need to impose the Bianchi identity $dF^I = 0$ and Maxwell equations (2.8). Substituting the field strengths (2.21) into the Bianchi identities $dF^I = 0$ gives

$$d\Theta^I = 0\,, \tag{2.25}$$

so the Θ^I are harmonic self-dual 2-forms on the base. The Maxwell equations (2.8) reduce to

$$\nabla^2\left(f^{-1}X_I\right) = \frac{1}{6}C_{IJK}\left(\Theta^J \,.\, \Theta^K\right), \tag{2.26}$$

where ∇^2 denotes the laplacian on the hyper-Kähler base B; and contracting (2.26) with X^I we obtain

$$\nabla^2 f^{-1} = -\frac{1}{3}Q_{IJ}\left(\left(\Theta^I \,.\, \Theta^J\right) + 2f^{-1}\left(dX^I \,.\, dX^J\right)\right) + \frac{2}{3}\left(G^+ \,.\, G^+\right), \tag{2.27}$$

where we have used the convention that for p-forms α, β on B, we set

$$(\alpha \,.\, \beta) = \frac{1}{p!}\alpha_{m_1\ldots m_p}\beta^{m_1\ldots m_p}\,. \tag{2.28}$$

The integrability conditions for the existence of a Killing spinor guarantee that the Einstein equation and scalar equations of motion are satisfied as a consequence of the above equations.

[1] Note that there is an additional factor of i compared with the expression given in [3] due to the change of signature of the metric.

2.3 Black hole solutions

Before proceeding with the uniqueness proof, it is useful to recall the form of the black hole solution found in [24] and [25]. This solution effectively extends the BMPV solution of the minimal ungauged five-dimensional supergravity to include arbitrary many abelian vector multiplets. In fact, just as for the BMPV solution, the solution does not only describe a single stationary black hole, but can be generalized to describe a multi-centred system of black holes with arbitrary positions. Here, we shall only consider the single-centre solutions. In our formalism, the black hole solution is obtained by taking the base space to be $B = \mathbb{R}^4$ equipped with metric

$$ds_4{}^2 = d\rho^2 + \frac{\rho^2}{4}\left[(\sigma_1^R)^2 + (\sigma_2^R)^2 + (\sigma_3^R)^2\right], \tag{2.29}$$

where σ_i^R are left-invariant 1-forms on SU(2) given in terms of the Euler angles θ, ϕ, ψ by

$$\begin{aligned}\sigma_1^R &= -\sin\psi d\theta + \cos\psi \sin\theta d\phi \\ \sigma_2^R &= \cos\psi d\theta + \sin\psi\sin\theta d\phi \\ \sigma_3^R &= d\psi + \cos\theta d\phi\end{aligned} \tag{2.30}$$

for $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$, $0 \le \psi < 4\pi$. Positive orientation is taken with respect to $\frac{\rho^3}{8} d\rho \wedge \sigma_1^R \wedge \sigma_2^R \wedge \sigma_3^R$. In addition, we take $\Theta^I = G^+ = 0$. Hence the equations (2.26) simplify to

$$\nabla^2(f^{-1}X_I) = 0\,, \tag{2.31}$$

where ∇^2 denotes the laplacian on $\mathbb{R}^4$. These equations are solved by taking

$$f^{-1}X_I = \nu_I + \frac{\mu_I}{\rho^2} \tag{2.32}$$

for constants μ_I, ν_I and hence f is given by

$$\begin{aligned}f^{-3} &= \frac{9}{2}C^{IJK}(1+\frac{\mu_I}{\rho^2})(1+\frac{\mu_J}{\rho^2})(1+\frac{\mu_K}{\rho^2}) \\ &= \alpha_0 + \frac{\alpha_1}{\rho^2} + \frac{\alpha_2}{\rho^4} + \frac{\alpha_3}{\rho^6}\,,\end{aligned} \tag{2.33}$$

where $C^{IJK} = \delta^{II'}\delta^{JJ'}\delta^{KK'}C_{I'J'K'}$ and

$$\begin{aligned}\alpha_0 &= \frac{9}{2}C^{IJK}\nu_I\nu_J\nu_K \\ \alpha_1 &= \frac{27}{2}C^{IJK}\mu_I\nu_J\nu_K \\ \alpha_2 &= \frac{27}{2}C^{IJK}\mu_I\mu_J\nu_K \\ \alpha_3 &= \frac{9}{2}C^{IJK}\mu_I\mu_J\mu_K\,,\end{aligned} \tag{2.34}$$

where we have made use of the identity

$$X^I = \frac{9}{2}C^{IJK}X_JX_K\,. \tag{2.35}$$

We require that $f > 0$ for $\rho > 0$, hence we must take $\alpha_0 \geq 0$ and $\alpha_3 \geq 0$; and to obtain an asymptotically flat solution we require $\alpha_0 > 0$. By rescaling t, we can without loss of generality set $\alpha_0 = 1$.

Lastly, we require that $d\omega$ be anti-self-dual on $\mathbb{R}^4$, so we set

$$\omega = \frac{j}{2\rho^2}\sigma_3^R\,, \tag{2.36}$$

for constant j. Observe that in order for the closed timelike curves to lie strictly within the horizon, we require that $f^{-3} - j^2\rho^{-6} > 0$ for $\rho > 0$, hence, in particular, $j^2 < \alpha_3$.

Note that the near-horizon geometry of the above black hole solutions is given by taking

$$f^{-1}X_I = \frac{\mu_I}{\rho^2} \tag{2.37}$$

with

$$f^{-3} = \frac{\alpha_3}{\rho^6} \tag{2.38}$$

and ω is given by (2.36).

3. Black hole uniqueness

In order to construct a uniqueness proof, we shall follow the methodology set out in [22]. In particular, we first show that the near-horizon geometry of a solution with a regular horizon is locally isometric to either flat space, $AdS_3 \times S^2$ or the under-rotating near-horizon BMPV solution. Then, making use of the fact that the base space is hyper-Kähler, the global properties of the solution which has near-horizon BMPV for its near-horizon geometry are investigated.

3.1 The near horizon geometry

Following the reasoning set out in [22, section 3.3], we shall introduce gaussian null co-ordinates u, r, x^A for $A = 1, 2, 3$ in a neighbourhood of the horizon, so that

$$V = \frac{\partial}{\partial u} \tag{3.1}$$

and

$$f = r\Delta\left(r, x^A\right) \tag{3.2}$$

with

$$ds^2 = -r^2\Delta^2 du^2 + 2dudr + 2rh_A du dx^A + \gamma_{AB}dx^A dx^B\,, \tag{3.3}$$

where $h_A = h_A(r, x^A)$, $\gamma_{AB} = \gamma_{AB}(r, x^A)$. Δ, h_A and γ_{AB} are smooth at the horizon; γ_{AB} defines a globally well-defined metric on a smooth riemannian 3-manifold in the near-horizon limit, and $\Delta > 0$ for $r > 0$. In the minimal theory, these assumptions were sufficient to ensure that the graviphoton gauge field strength is regular on the horizon. In the more general theory which we consider here, we shall see that regularity of the metric on the horizon is sufficient to prove that $X_I F^I$ is regular on the horizon. However, in order to construct the uniqueness proof we shall in fact assume a stronger condition on the fields, namely that all of the X^I and all components of the F^I are regular at $r = 0$ in the gaussian null co-ordinates.

Next, note that the three hyper-Kähler forms can be written as

$$J^i = dr \wedge Z^i + r\left(h \wedge Z^i - \Delta \star_3 Z^i\right), \tag{3.4}$$

where $\star_3$ denotes the hodge dual defined with respect to γ_{AB} and $Z^i = Z^i_A dx^A$, $h = h_A dx^A$. As the J^i satisfy the algebra of the imaginary unit quaternions, it is straightforward to show that the Z^i define an orthonormal basis on the 3-manifold H with metric γ_{AB}. The J^i are closed, so we obtain

$$\hat{d}Z^i = -\frac{1}{2}\partial_r(r\Delta)\epsilon_{ijk}Z^j \wedge Z^k + \partial_r(rh) \wedge Z^i - r\Delta\epsilon_{ijk}\partial_r Z^j \wedge Z^k + rh \wedge \partial_r Z^i \tag{3.5}$$

and

$$\star_3 \hat{d}h - \hat{d}\Delta - \Delta h + r\partial_r\Delta h - 2r\Delta\partial_r h - r\star_3(h \wedge \partial_r h) - r\Delta^2\epsilon_{ijk}Z^i\langle Z^j, \partial_r Z^k\rangle = 0\,, \tag{3.6}$$

where if Y is a p-form of the type

$$Y = \frac{1}{p!}Y_{A_1\ldots A_p}dx^{A_1} \wedge \cdots \wedge dx^{A_p} \tag{3.7}$$

we define

$$\hat{d}Y = \frac{1}{(p+1)!}(p+1)\partial_{[A_1}Y_{A_2\ldots A_{p+1}]}dx^{A_1} \wedge \cdots \wedge dx^{A_{p+1}}\,. \tag{3.8}$$

and $\langle , \rangle$ denotes the inner product on H with respect to the metric γ_{AB}. In addition, for $r > 0$ we obtain

$$\omega = -\frac{1}{r^2\Delta^2}dr - \frac{1}{r\Delta^2}h \tag{3.9}$$

and hence

$$G^+ = dr \wedge \mathcal{G} + r(h \wedge \mathcal{G} + \Delta \star_3 \mathcal{G})\,, \tag{3.10}$$

where

$$\mathcal{G} = -\frac{3}{2r\Delta^2}\hat{d}\Delta + \frac{3}{2\Delta^2}\partial_r\Delta h - \frac{3}{2\Delta}\partial_r h - \frac{1}{2}\epsilon_{ijk}Z^i\left\langle Z^j, \partial_r Z^k\right\rangle\,. \tag{3.11}$$

All of the above identities are identical to those found in [22] for the minimal theory. To extend the near-horizon classification to the more general theory, it is convenient to set

$$\Theta^I = -\frac{2}{3}X^I G^+ + N^I \tag{3.12}$$

so that $X_I N^I = 0$. Setting

$$N^I = dr \wedge T^I + r(h \wedge T^I + \Delta \star_3 T^I) \tag{3.13}$$

for $T^I = T^I_A dx^A$, and using (2.21) we find

$$\begin{aligned} F^I = & \left[\partial_r(r\Delta X^I)dr + r\hat{d}(\Delta X^I)\right] \wedge du + dr \wedge Q^I + rh \wedge Q^I + \\ & + r\Delta \star_3 T^I - X^I \star_3 h - rX^I \star_3 \partial_r h - \frac{2}{3}\Delta r\epsilon_{ijk} \star_3 Z^i\left\langle Z^j, \partial_r Z^k\right\rangle, \end{aligned} \tag{3.14}$$

where

$$Q^I = T^I + \frac{1}{r\Delta}\hat{d}X^I - \frac{1}{\Delta}\partial_r X^I h + \frac{1}{3}X^I \epsilon_{ijk} Z^i \left\langle Z^j, \partial_r Z^k \right\rangle . \tag{3.15}$$

On using $X_I T^I = 0$, $X_I \hat{d}X^I = 0$ and $X_I \partial_r X^I = 0$ we find that $X_I F^I$ is regular on the horizon. However, our assumption of the regularity of both X^I and F^I is somewhat stronger. In particular we find that Q^I is regular on the horizon. Note that the spatial components of F^I are given by

$$\begin{aligned}\frac{1}{2}F^I_{AB} dx^A \wedge dx^B &= r\Delta \star_3 Q^I - \star_3 \hat{d}X^I + r\partial_r X^I \star_3 h - X^I \star_3 h - \\ &\quad - rX^I \star_3 \partial_r h - \frac{2}{3}\Delta r \epsilon_{ijk} \star_3 Z^i \left\langle Z^j, \partial_r Z^k \right\rangle . \end{aligned}\tag{3.16}$$

Hence, evaluating the spatial components of the Bianchi identity at $r = 0$ we find

$$\hat{d}\left(X^I \star_3 h + \star_3 \hat{d}X^I\right) = O(r) . \tag{3.17}$$

Contracting this expression with X_I we obtain

$$\hat{d}\star_3 h + \frac{2}{3} Q_{IJ} \hat{d}X^I \wedge \star_3 \hat{d}X^J = O(r) . \tag{3.18}$$

Integrating over H we find that

$$\int_H Q_{IJ} \hat{d}X^I \wedge \star_3 \hat{d}X^J = O(r) \tag{3.19}$$

and as Q_{IJ} defines a positive-definite inner product it follows that

$$\hat{d}X^I = O(r) \tag{3.20}$$

and hence

$$\hat{d}\star_3 h = O(r) . \tag{3.21}$$

So, it follows that h, Δ and the Z^i satisfy exactly the same constraints on the horizon as in the minimal theory. These constraints were analysed in detail in [22] so we shall only present the results of that analysis here.

In particular, it is straightforward to see that Δ must be constant on the horizon. The case for which $\Delta = 0$ on the horizon is special. There are then two sub-cases; in the first $h \neq 0$ on the horizon and the near-horizon geometry is locally isometric to $AdS_3 \times S^2$. It was originally argued in the [22, appendix] that solutions with this near-horizon geometry can be excluded; however, more recent work in [26] has shown that there indeed exist asymptotically flat supersymmetric black ring solutions with $AdS_3 \times S^2$ near-horizon geometry. In the second sub-case, $h = 0$ on the horizon and the near-horizon geometry is locally isometric to Minkowski space; it has not yet been determined whether this geometry can arise as the near-horizon geometry of a black hole.

Notwithstanding this difficulty, we shall assume henceforth that $\Delta > 0$ on the horizon. Then from [22, section 3.6] it is straightforward to see that local co-ordinates ϕ, θ, ψ' can be introduced with respect to which the metric on H can be written as

$$ds_3^2 = \frac{\mu}{4}\left[\left(1 - \frac{j^2}{\mu^3}\right)(d\psi' + \cos\theta d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2\right] \tag{3.22}$$

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with

$$h = -j\mu^{-3/2}\sqrt{1-\frac{j^2}{\mu^3}}(d\psi' + \cos\theta d\phi) \tag{3.23}$$

where μ and j are constants with $\mu > 0$, $j^2 < \mu^3$ and

$$\Delta = 2\mu^{-1/2}\sqrt{1-\frac{j^2}{\mu^3}}\,. \tag{3.24}$$

To summarize, we have shown that in the near-horizon limit, X^I and Δ are constant. Assuming that $\Delta \neq 0$ on the horizon, we have proven that the near-horizon geometry is locally isometric to that of the near-horizon BMPV solution in the minimal theory. This is also the near-horizon geometry of the Chamseddine and Sabra black holes. In particular, if $j = 0$ then H is locally isometric to the round 3-sphere. Globally, we must have $H = S^3/\Gamma$ where Γ is a discrete subgroup of $\mathrm{SU}(2)_R$. If $j \neq 0$ then H is locally isometric to a squashed 3-sphere; globally $H = S^3/\Gamma$ where Γ is a cyclic group.

3.2 Global analysis

Given the similarity between the analysis of the near-horizon geometries in the general and the minimal five-dimensional supergravity theories, it is unsurprising that there is also a considerable similarity in the global analysis of the black hole solutions for which the near-horizon geometry is the near-horizon BMPV geometry. In particular, if we make a change of co-ordinates

$$\begin{aligned} R &= (2r)^{1/2}\mu^{1/4}\left(1-\frac{j^2}{\mu^3}\right)^{1/4} \\ \psi &= \psi' - 2j\mu^{-3/2}\left(1-\frac{j^2}{\mu^3}\right)^{-1/2}\log R \end{aligned} \tag{3.25}$$

then locally, in a neighbourhood of the horizon, the metric on the hyper-Kähler base takes the form

$$\begin{aligned} ds_4^2 = dR^2 + \frac{R^2}{4}&\left[(d\psi+\cos\theta d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2\right] + \\ &+ O(R^2)dR^2 + O(R^3)dRdy^A + O(R^4)dy^A dy^B\,, \end{aligned} \tag{3.26}$$

where $y^A = (\psi, \theta, \phi)$. Hence, locally, the base space is flat. However, it is, in principle, possible for there to be a conical singularity at $R = 0$. In fact, just as in the case of the minimal theory [22], the conical singularity must be an $A-D-E$ orbifold singularity because the base is hyper-Kähler. Moreover, we also require that the solution be asymptotically flat, i.e. f must tend to a positive constant at infinity, and ω must decay sufficiently rapidly so that the ADM angular momentum is well-defined. Hence the base space must be asymptotically euclidean. Now, the $A-D-E$ orbifold singularities at $R = 0$ can be resolved [27] by blowing up the singularity. Thus we obtain a new hyper-Kähler base space which is complete and asymptotically euclidean. It is known that only one such manifold exists [28]; the base must be globally $\mathbb{R}^4$ equipped with metric

$$ds_4^2 = d\rho^2 + \frac{\rho^2}{4}\left[(d\psi + \cos\theta d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2\right] \tag{3.27}$$

and

$$R = \rho + O(\rho^3)\,, \tag{3.28}$$

where $0 \le \theta \le \pi$ and $\phi \sim \phi + 2\pi$, $\psi \sim \psi + 4\pi$.

The next step in the global analysis is to show that $\Theta^I = 0$. To see this, note first that

$$F^I = d(r\Delta X^I) \wedge du - d\left(\frac{X^I}{r\Delta}\right) \wedge dr - d\left(\frac{X^I}{\Delta}h\right) + \Theta^I\,. \tag{3.29}$$

Moreover, we recall that as X^I and Δ are constant on the horizon (with $\Delta \neq 0$ on the horizon), it follows that the $d(\frac{X^I}{r\Delta}) \wedge dr$ term in this expression is regular on the horizon. Hence, as we assume that F^I is regular at the horizon, Θ^I (and hence also G^+) is regular at the origin of the base space. Moreover, G^+ must vanish at infinity in $\mathbb{R}^4$ due to the asymptotic decay of ω. In addition, f tends to a positive constant at infinity with a decay sufficient to ensure the existence of a well-defined ADM mass. Using these facts, it is straightforward to see from (2.27) that $Q_{IJ}((\Theta^I \,.\, \Theta^J) + 2f^{-1}(dX^I \,.\, dX^J))$ must vanish at infinity. Hence, at infinity the Θ^I must vanish and the X^I are constant. As Θ^I is closed and globally defined on $\mathbb{R}^4$ it follows that there exists a 1-form Λ^I also globally defined on $\mathbb{R}^4$, with Λ^I vanishing at infinity, such that $\Theta^I = d\Lambda^I$. Then

$$0 = \int_{S^3_\infty} \Lambda^I \wedge \Theta^I = \int_{\mathbb{R}^4} \Theta^I \wedge \Theta^I = \int_{\mathbb{R}^4} (\Theta^I \,.\, \Theta^I)\,, \tag{3.30}$$

where S^3_∞ denotes the 3-sphere at infinity in $\mathbb{R}^4$. Hence it follows that $\Theta^I = G^+ = 0$.

As $\Theta^I = 0$, we find from (2.26) that the $f^{-1}X_I$ are harmonic on $\mathbb{R}^4$. Suppose that $X_I \to \mu^{-1}\mu_I$ as $R \to 0$ and $X_I \to \nu_I$ as $R \to \infty$ for constants μ_I, ν_I. Then in a neighbourhood of the origin we have $f \sim \rho^2/\mu$, and hence

$$f^{-1}X_I = \frac{\mu_I}{\rho^2} + g_I\,, \tag{3.31}$$

where g_I is $O(\rho^0)$ near $\rho = 0$. As $f^{-1}X_I$ is regular outside the horizon, we note that g_I must be regular, bounded and harmonic on $\mathbb{R}^4$. Hence g_I is constant. By re-scaling u we can without loss of generality set $f \to 1$ as $R \to \infty$, so we must have $g_I = \nu_I$. Observe that (3.31) implies that

$$\begin{aligned} f^{-3} &= \frac{9}{2}C^{IJK}\left(1 + \frac{\mu_I}{\rho^2}\right)\left(1 + \frac{\mu_J}{\rho^2}\right)\left(1 + \frac{\mu_K}{\rho^2}\right) \\ &= \alpha_0 + \frac{\alpha_1}{\rho^2} + \frac{\alpha_2}{\rho^4} + \frac{\alpha_3}{\rho^6}\,, \end{aligned} \tag{3.32}$$

where

$$\begin{aligned}
\alpha_0 &= \frac{9}{2} C^{IJK} \nu_I \nu_J \nu_K \\
\alpha_1 &= \frac{27}{2} C^{IJK} \mu_I \nu_J \nu_K \\
\alpha_2 &= \frac{27}{2} C^{IJK} \mu_I \mu_J \nu_K \\
\alpha_3 &= \frac{9}{2} C^{IJK} \mu_I \mu_J \mu_K \,.
\end{aligned} \tag{3.33}$$

Hence we must have $\alpha_0 = 1$ and $\alpha_3 = \mu^3$. Observe that $j^2 < \alpha_3$.

Finally, we note that by exactly the same reasoning as set out in [22, section 3.6], we must have

$$\omega = \frac{j}{2\rho^2} \sigma_3^R + db \tag{3.34}$$

for some function b. As b can be absorbed into the time co-ordinate t by a shift, we can set without loss of generality

$$\omega = \frac{j}{2\rho^2} \sigma_3^R \,. \tag{3.35}$$

Hence we have proven that the only regular black hole solution of this theory with a near-horizon geometry which is the near-horizon BMPV geometry (in which limit the F^I and X^I are sufficiently regular) is the under-rotating (i.e. $j^2 < \mu^3$) black hole solution of Chamseddine and Sabra [24, 25].

4. Conclusions

In this paper we have presented an extension of the black hole uniqueness theorem of [22] to a more general ungauged five-dimensional supergravity coupled to arbitrary many abelian vector multiplets. It is remarkable that the Very Special geometry associated with this theory imposes sufficiently strong constraints on supersymmetric solutions to allow for such a theorem to be proved. The only sufficiently regular solutions of the theory which have an event horizon locally isometric to the near-horizon BMPV geometry are the black holes found in [24] and [25]. It would be interesting to determine whether or not the near-horizon solutions with $\Delta = 0$ and $h = 0$ on the horizon can arise as the near-horizon geometry of an asymptotically flat supersymmetric black hole. Also, in order to prove the theorem, rather strong regularity conditions were imposed on the scalars and field strengths at the horizon. It might be possible to weaken these conditions and investigate whether additional black hole solutions exist.

Another interesting problem would be to attempt to prove a black hole uniqueness theorem in gauged five-dimensional supergravity. Recently, the first examples of regular supersymmetric black holes of such a theory have been found in [23] and [29]. However, even for the case of the minimal gauged theory, attempting to adapt the methods used in the ungauged uniqueness theorem is somewhat problematic. In particular, for the near-horizon analysis, it is by no means apparent that the scalar Δ must be constant on the horizon. More seriously, the base space in the gauged theory is not hyper-Kähler, rather it

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is only Kähler. Hence the global analysis which was used for the ungauged theory cannot be straightforwardly generalized. Finally, we remark that there are more general five-dimensional supergravity theories; a recent useful discussion of this can be found in [30]. It would be useful to determine if these more general theories admit new black hole solutions.

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